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SUBCLASS OF MULTIVALENT FUNCTIONS DEFINED BY GENERALIZED BERNARDI-LIBERA-LIVINGSTON INTEGRAL OPERATOR

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Abstract

By means of certain differential operator we introduce and investigate a subclass$\Re A_{n,m}^{q}(\lambda , b , \delta)$of q-valently analytic functions. The various results obtained here for this class .we have attempted to obtain radius of starlikeness, convexity and closure theorem for the class$\Re A_{n,m}^{q}(\lambda , b , \delta)$

Keywords & Phrases :- Multivalent function, radius of star likeness , integral operator.

1 INTRODUCTION

This chapter introduces p-valent functions and its various properties. By means of certain Generalized Bernardi-Libera-Livingston Integral Operator,we introduce and investigate new subclass $\Re A_{n,m}^{p}(\lambda , b , \delta)$ of p-valently analytic functions of complex order. The various results obtained here for this subclass.

Let $A(n)$ denote the class of functions $f(z)$ normalized by

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k , \qquad \ldots\ldots\ldots \tag{1}$$

$$a_k \geq 0 \quad and \ n , p \in \mathbb{N} = \{1,2,3, \ldots .\}$$

which are analytic and p-valent in the open unit disc $E = \{z : z \in \mathbb{C} \ and \mid z \mid < 1 \}$

We introduce here Generalized Bernardi-Libera-Livingston Integral Operator:

$$\mathcal{F}_p^{\lambda} f(z) = \frac{\lambda + p}{z^{\lambda}} \int_0^z x^{\lambda - 1} f(x) dx \quad , \quad (\lambda > -p \, ; z \in U \ )$$

Simplifying we get

$$\mathcal{F}_p^{\lambda} f(z) = z^p - \sum_{k=n+p}^{\infty} \frac{\lambda + p}{\lambda + k} a_k z^k ,$$

$$(\mathcal{F}_p^{\lambda} f(z))^{(m)} = \binom{p}{m} z^{p-m} - \sum_{k=n+p}^{\infty} \binom{k}{m} \left(\frac{\lambda + p}{\lambda + k}\right) a_k z^{k-m}$$

Where $\binom{k}{m} = \frac{k(k-1)(k-2)......(k-m+1)}{m!}$

By using the operator $\mathcal{F}_p^{\lambda} f(z)$, we introduce new subclass $\Re A_{n,m}^{p}(\lambda , b , \delta)$ of p-valently analytic function $f(z)$ satisfying the following inequality

$$\left| \frac{1}{b} \left(\frac{\delta z \left(\mathcal{F}_p^{\lambda} f(z)\right)^{(m+1)} + \lambda z^2 \left(\mathcal{F}_p^{\lambda} f(z)\right)^{(m+2)}}{\lambda z \left(\mathcal{F}_p^{\lambda} f(z)\right)^{(m+1)} + (\delta - \lambda) \left(\mathcal{F}_p^{\lambda} f(z)\right)^{(m)}} - (p - m) \right) \right| < 1 \dots \dots (2)$$

$p \in \mathbb{N}$, $m \in \mathbb{N} \, U \{0\}, z \in E$, $p > \max(m, -\lambda)$, $b \in \mathbb{C} \, U \{0\}$, $\lambda \geq 0 , 0 < \delta \leq 1$

The objective of thispaper is to investigate the various properties and characteristics of analytic p-valent functions belonging to the subclass$\Re A_{n,m}^{p}(\lambda , b , \delta)$ which we have defined here. Apart from deriving a set of coefficient bounds for each of these function classes, we establish radius of starlikeness, convexity and closure theorem.Our definitions of the function class $\Re A_{n,m}^{p}(, b , \delta)$ are motivated by the investigation of H. M. Srivastava and others [2], we have relaxed the parametric constraint $0 \leq \lambda \leq 1$.

THEOREM 1:- A function $f(z) \in A(n)$ and defined by

$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$, $a_k \geq 0$ $\quad and \quad p \in \mathbb{N}$, is in $\Re A_{n,m}^{p}(\lambda , b , \delta)$ if and only if

$$\sum_{k=n+p}^{\infty} \left(\frac{\lambda + p}{\lambda + k}\right) \binom{k}{m} [\lambda(k - m - 1) + \delta)][k - p + |b|] a_k$$

$$\leq |b| \binom{p}{m} [\lambda(p - m - 1) + \delta]$$

COROLLARY 1:- $f(z) \in \Re A_{n,m}^{p}(\lambda , b , \delta)$ then

$$a_k \leq \frac{|b| \binom{p}{m} [\lambda(p - m - 1) + \delta]}{\binom{k}{m} \left(\frac{\lambda+p}{\lambda+k}\right) [\lambda(k - m - 1) + \delta][k - p + |b|]}$$

COROLLARY 2:- for $p = 1 , m = 0$ we have

$$a_k \leq \frac{|b| \delta(\lambda + k)}{(\lambda + 1)[\lambda(k - 1) + \delta][k - 1 + |b|]} \quad , k \geq n + p$$

COROLLARY 3:- for $p = 1 , m = 1$ we have

$$a_k \leq \frac{|b|[\delta - \lambda](\lambda + k)}{k(\lambda + 1)[\lambda k + \delta][k - 1 + |b|]}$$

2. RADII OF CLOSE-TO-CONVEXITY, STARLIKENESS AND CONVEXITY

A function $f(z) \in A(n)$ is said to be close to convex of order $\propto$ $(0 \leq \propto < 1)$ if

$Re\left\{f'(z)\right\} > \propto$for all $z \in E$

A function $f(z) \in A(n)$ is said to be starlike of order $\propto$ $(0 \leq \propto < 1)$ if

$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \propto$for all $z \in E$

A function $f(z) \in A(n)$ is said to be convex of order $\propto$ $(0 \leq \propto < 1)$if and only if $zf'(z)$ is starlike of order $\propto$, that is

$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \propto$for all $z \in E$

THEOREM 2.1:- If$f(z) \in \Re A_{n,m}^{p}(\lambda , b , \delta)$, then f is close to convex of order $\propto$ in $|z| < r_1(p, n, m, \lambda , b , \delta , \propto)$where

$$r_1 = \inf_k \left(\left(\left(\frac{\lambda + p}{\lambda + k}\right)\binom{k}{m}\frac{(p-\propto)[\lambda(k - m - 1) + \delta)][k - p + |b|]}{k|b|\binom{p}{m}[\lambda(p - m - 1) + \delta]}\right)^{\frac{1}{k-p}}\right)$$

PROOF:- It is sufficient to show that $\left|\frac{f'(z)}{z^{p-1}} - p\right| < p-\propto$

$$f'(\mathbf{z}) = pz^{p-1} - \sum_{k=n+p}^{\infty} k\, a_k\, z^{k-1}$$

$$\frac{f'(z)}{z^{p-1}} = p - \sum_{k=n+p}^{\infty} k\, a_k\, z^{k-p}$$

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq \sum_{k=n+p}^{\infty} k|a_k||z|^{k-p} < p-\propto \quad \ldots\ldots\ldots\ldots \quad (2.1.1)$$

we have

$$\sum_{k=n+p}^{\infty} \left(\frac{\lambda + p}{\lambda + k}\right)\binom{k}{m}[\lambda(k - m - 1) + \delta)][k - p + |b|]a_k \leq |b|\binom{p}{m}[\lambda(p - m - 1) + \delta]$$

That is $\sum_{k=n+p}^{\infty}\left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m}\frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]} a_k \leq 1$ ……………….(2.1.2)

Observe that (2.1.1) is true if

$$\frac{k|z|^{k-p}}{p-\propto} \leq \left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m}\frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}$$

Therefore

$$|z| \leq \left(\left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m}\frac{(p-\propto)[\lambda(k-m-1)+\delta)][k-p+|b|]}{k|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}\right)^{\frac{1}{k-p}} ,$$

$(p \neq k , p, k \in \mathbb{N})$, which complete the proof.

THEOREM 2.2 :- If $f(z) \in \Re A_{n,m}^{p}(\lambda , b , \delta)$, then f is starlike of order $\propto$ in $|z| < r_2(p, n, m, \lambda , b , \delta , \propto)$ where

$$r_2 = \inf_{k}\left(\left(\left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m}\frac{(p-\propto)[\lambda(k-m-1)+\delta)][k-p+|b|]}{(k-\propto)|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}\right)^{\frac{1}{k-p}}\right)$$

PROOF:- We must show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p-\propto$$

$$zf'(z) - p\,f(z) = pz^p - \sum_{k=n+p}^{\infty} ka_k\, z^k - pz^p + p\sum_{k=n+p}^{\infty} a_k\, z^k = -\sum_{k=n+p}^{\infty}(k-p)a_k\, z^k$$

We have

$$\left|\frac{zf'(z)}{f(z)} - p\right| = \left|\frac{-\sum_{k=n+p}^{\infty}(k-p)a_k z^k}{z^p-\sum_{k=n+p}^{\infty}a_k z^k}\right| \leq \frac{\sum_{k=n+p}^{\infty}(k-p)|a_k|\,|z|^{k-p}}{1-\sum_{k=n+p}^{\infty}|a_k|\,\,|z|^{k-p}} \leq p-\propto$$

……………(2.1.3)

Hence (2.1.3) holds true if

$\sum_{k=n+p}^{\infty}(k-p)|a_k|\,|z|^{k-p} \leq (p-\propto)\,(1-\sum_{k=n+p}^{\infty}|a_k|\,|z|^{k-p})$

Or equivalently

$\sum_{k=p+1}^{\infty}\frac{(k-\propto)}{(p-\propto)}|a_k|\,|z|^{k-p} \leq 1$ ……….. (2.1.4)

we have

$$\sum_{k=n+p}^{\infty}\left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m}\frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}a_k \leq 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots.(2.1.5)$$

Hence by using (2.1.4) and (2.1.5) we get

$$\frac{(k-\propto)}{(p-\propto)}|z|^{k-p} \le \left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m}\frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}$$

$$|z|^{k-p} \le \left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m}\frac{(p-\propto)[\lambda(k-m-1)+\delta)][k-p+|b|]}{(k-\propto)|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}$$

$$|z| \le \left(\left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m}\frac{(p-\propto)[\lambda(k-m-1)+\delta)][k-p+|b|]}{(k-\propto)|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}\right)^{\frac{1}{k-p}}$$

$(p \neq k, p, k \in \mathbb{N})$, which complete the proof.

THEOREM 2.3:- If$(z) \in \Re A^{p}_{n,m}(\lambda, b, \delta)$, then f is convex of order $\propto$ in $|z| < r_3(p, n, m, \lambda, b, \delta, \propto)$where

$$r_3 = \inf_{k}\left(\left(\left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m}\frac{p(p-\propto)[\lambda(k-m-1)+\delta)][k-p+|b|]}{k(k-\propto)|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}\right)^{\frac{1}{k-p}}\right)$$

PROOF:-We know that f is convex if and only if zf' is starlike

We must show that

$$\left|\frac{zg'(z)}{g(z)} - p\right| \le p-\propto$$

Where $g(z) = zf'(z)$

$$g(z) = p\,z^p - \sum_{k=n+p}^{\infty} k\,a_k\,z^k$$

$$zg'(z) = p^2\,z^p - \sum_{k=n+p}^{\infty} k^2\,a_k\,z^k$$

$$zg'(z) - p\,g(z) = p^2 z^p - \sum_{k=n+p}^{\infty} k^2 a_k\,z^k - p^2 z^p + p\sum_{k=n+p}^{\infty} k\,a_k\,z^k$$

$$= -\sum_{k=n+p}^{\infty} k(k-p)\,a_k\,z^k$$

$$\left|\frac{zg'(z)}{g(z)} - p\right| = \left|\frac{-\sum_{k=n+p}^{\infty} k(k-p)\,a_k\,z^k}{p\,z^p - \sum_{k=n+p}^{\infty} k\,a_k\,z^k}\right| \le \frac{\sum_{k=n+p}^{\infty} k(k-p)|a_k|\,|z|^{k-p}}{p - \sum_{k=n+p}^{\infty} k\,|a_k|\,|z|^{k-p}} \le p-\propto$$

Therefore we have

$$\sum_{k=n+p}^{\infty} k(k-p)|a_k|\,|z|^{k-p} \leq (p-\propto)[p - \sum_{k=n+p}^{\infty} k\,|a_k|\,|z|^{k-p}]$$

$\sum_{k=n+p}^{\infty} \frac{k(k-\propto)}{p(p-\propto)} |a_k||z|^{k-p} \leq 1$ ... ...(2.1.6)

we have

$$\sum_{k=n+p}^{\infty} \left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m}\frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]} a_k \leq 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.1.7)$$

Hence by using (2.1.6) and (2.1.7) we get

$$\frac{k(k-\propto)}{p(p-\propto)}|z|^{k-p} \leq \left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m}\frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}$$

$$|z| \leq \left(\left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m}\frac{p(p-\propto)[\lambda(k-m-1)+\delta)][k-p+|b|]}{k(k-\propto)|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}\right)^{\frac{1}{k-p}}$$

$(p \neq k, p, k \in \mathbb{N})$, which complete the proof.

3. CLOSURE THEOREM

THEOREM 3.1 :

Let $f_1(z) = z^p$ and $f_k(z) = z^p - \left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m}\frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]} z^k$ for $k \geq n + p$

Then $f(z) \in \Re A_{n,m}^{p}(\lambda, b, \delta)$ if and only if $f(z)$ can be expressed in the form

$f(z) = \lambda_1 f_1(z) + \sum_{k=n+p}^{\infty} \lambda_k f_k(z)$ where $\lambda_k \geq 0$ and $\lambda_1 + \sum_{k=n+p}^{\infty} \lambda_k = 1$

PROOF: - Suppose $f(z)$ can be expressed in the form

$$f(z) = \lambda_1 f_1(z) + \sum_{k=n+p}^{\infty} \lambda_k f_k(z)$$

$$= \lambda_1 z^p + \sum_{k=n+p}^{\infty} \lambda_k \left[z^p - \left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m}\frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]} z^k\right]$$

$$= [\lambda_1 + \sum_{k=n+p}^{\infty} \lambda_k] z^p - \sum_{k=n+p}^{\infty} \lambda_k \left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m}\frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]} z^k$$

$$= z^p - \sum_{k=n+p}^{\infty} \lambda_k \left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m} \frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]} z^k$$

Then

$$\sum_{k=n+p}^{\infty} \lambda_k \left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m} \frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]} \cdot \frac{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}{\left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m}[\lambda(k-m-1)+\delta)][k-p+|b|]} \ z^k$$

$$= \sum_{k=n+p}^{\infty} \lambda_k = 1 - \lambda_1 \le 1$$

Therefore $f(z) \in \Re A_{n,m}^{p}(\lambda, b, \delta)$

Conversely, suppose that $f(z) \in \Re A_{n,m}^{p}(\lambda, b, \delta)$

We have

$$a_k \le \left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m} \frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}$$

We take

$$\lambda_k = \frac{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}{\left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m}[\lambda(k-m-1)+\delta)][k-p+|b|]} a_k$$

$k \ge n+p$ and $\sum_{k=n+p}^{\infty} \lambda_k = 1 - \lambda_1$

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$$

$$= z^p - \sum_{k=n+p}^{\infty} \left(\frac{\lambda+p}{\lambda+k}\right)\binom{k}{m} \frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]} z^k$$

$$= z^p - \sum_{k=n+p}^{\infty} \lambda_k \, [z^p - f_k(z)] = z^p \left[1 - \sum_{k=n+p}^{\infty} \lambda_k\right] - \sum_{k=n+p}^{\infty} \lambda_k f_k(z)$$

$$= \lambda_1 f_1(z) + \sum_{k=n+p}^{\infty} \lambda_k f_k\,(z)$$

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